

Dominating Sets inducing Large Components in Maximal Outerplanar Graphs

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Abstract

For a maximal outerplanar graph G of order n at least 3, Matheson and Tarjan showed that G has domination number at most $n/3$. Similarly, for a maximal outerplanar graph G of order n at least 5, Dorfling, Hattingh, and Jonck showed, by a completely different approach, that G has total domination number at most $2n/5$ unless G is isomorphic to one of two exceptional graphs of order 12.

We present a unified proof of a common generalization of these two results. For every positive integer k , we specify a set \mathcal{H}_k of graphs of order at least $4k + 4$ and at most $4k^2 - 2k$ such that every maximal outerplanar graph G of order n at least $2k + 1$ that does not belong to \mathcal{H}_k has a dominating set D of order at most $\lfloor \frac{kn}{2k+1} \rfloor$ such that every component of the subgraph $G[D]$ of G induced by D has order at least k .

Keywords: domination; total domination; maximal outerplanar graph

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1 Introduction

The two most prominent domination parameters [5, 6], the domination number $\gamma(G)$ and the total domination number $\gamma_t(G)$ of a graph G , have both been studied in detail for maximal outerplanar graphs [1–3, 9]. Two fundamental results in this context are as follows.

Theorem 1 (Matheson and Tarjan [7]) *If G is a maximal outerplanar graph of order n at least 3, then $\gamma(G) \leq \lfloor \frac{n}{3} \rfloor$.*

Theorem 2 (Dorfling, Hattingh, and Jonck [4]) *If G is a maximal outerplanar graph of order n at least 5 that is not isomorphic to one of the two graphs H_1 and H_2 in Figure 1, then $\gamma(G) \leq \lfloor \frac{2n}{5} \rfloor$.*

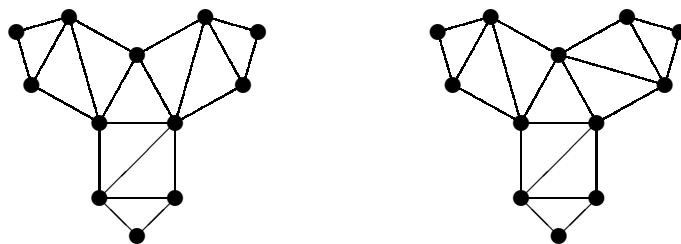


Figure 1: The two exceptional graphs H_1 and H_2 for Theorem 2.

The proofs of these two results in [4, 7] are quite different. While Theorem 1 follows from an elegant labeling argument, the proof of Theorem 2 relied on a detailed case analysis; one reason for this difference probably being the existence of the two exceptional graphs.

Our goal in the present paper is a unified proof of a common generalization of these two results.

For some positive integer k and a graph G , a set D of vertices of G is a *k -component dominating set* of G if every vertex in $V(G) \setminus D$ has a neighbor in D , and every component of the subgraph $G[D]$ of G induced by D has order at least k . The minimum cardinality of a k -component dominating set of G is the *k -component domination number* $\gamma_k(G)$ of G .

Note that a graph has a k -component dominating set if and only if each of its components has order at least k . Clearly, $\gamma_1(G)$ coincides with the domination number of G , and $\gamma_2(G)$ coincides with the total domination number of G , respectively. The notation “ $\gamma_k(G)$ ” has already been used to denote various other domination parameters. We chose this notation for its simplicity, and because there is little danger of confusion within the context of this paper.

For every positive integer k , we will specify a set \mathcal{H}_k of graphs each of order at least $4k + 4$ and at most $4k^2 - 2k$ such that our main result reads as follows.

Theorem 3 *If k and n are positive integers with $n \geq 2k + 1$, and G is a maximal outerplanar graph of order n , then*

$$\gamma_k(G) \leq \begin{cases} \lceil \frac{kn}{2k+1} \rceil, & \text{if } G \in \mathcal{H}_k \\ \lfloor \frac{kn}{2k+1} \rfloor, & \text{otherwise.} \end{cases}$$

As we show below, the bound in Theorem 3 is actually tight for all values of k and n with $n \geq 2k + 1$. For $k = 1$, we have $4k + 4 > 4k^2 - 2k$, which implies that \mathcal{H}_1 is necessarily empty, that is, Theorem 3 implies Theorem 1. Furthermore, we will see that $\mathcal{H}_2 = \{H_1, H_2\}$, that is, Theorem 3 implies Theorem 2.

The rest of the paper is devoted to the proof of Theorem 3.

2 Results

For every maximal outerplanar graph, we will tacitly assume that it is embedded in the plane in such a way that all its vertices are on the boundary of the unbounded face. This implies that every bounded face is bounded by a triangle. Furthermore, we assume that subgraphs inherit their embeddings in the natural way.

Let G be a maximal outerplanar graph of order at least 3. The boundary of the unbounded face of G is a Hamiltonian cycle $C(G)$ of G . A chord of G is an edge of G that does not belong to $C(G)$. Adding a chord xy of G to $C(G)$ results in a graph that has exactly two cycles C_1 and C_2 that are distinct from $C(G)$. Furthermore, C_1 and C_2 are the boundaries of two maximal outerplanar subgraphs of G whose union is G and whose intersection is the edge xy . We will refer to these two graphs as the *subgraphs of G generated by xy* . We refer to the edges of some graph G as *G -edges*.

For positive integers s and t , let $[s, t]$ be the set of positive integers at least s and at most t , and let $[t] = [1, t]$.

For positive integers k and n with $n \geq \max\{3, k\}$, let

$$\gamma_k(n) = \max\{\gamma_k(G) : G \text{ is a maximal outerplanar graph of order } n\}.$$

Lemma 4 *If k, k' , and n are positive integers with $n \geq \max\{3, k\}$ and $k \geq k'$, then*

$$(i) \quad \gamma_k(n) \leq \gamma_k(n+1), \text{ and}$$

$$(ii) \quad \gamma_{k'}(n) \leq \gamma_k(n).$$

Proof: (i) Let G be a maximal outerplanar graph of order n such that $\gamma_k(G) = \gamma_k(n)$. For some $C(G)$ -edge uv of G , let G' arise from G by adding a new vertex x and the new edges ux and xv . Clearly, G' is a maximal outerplanar graph of order $n + 1$. Let D' be a minimum k -component dominating set of G' . If either $x \notin D'$ or $x \in D'$ and the component of $G'[D']$ that contains x has order at least $k + 1$, then $D' \setminus \{x\}$ is a k -component dominating set of G , which implies $\gamma_k(n) = \gamma_k(G) \leq \gamma_k(G') \leq \gamma_k(n + 1)$. Hence, we may assume that $x \in D'$, and that the component of $G'[D']$ that contains x has order exactly k . Since $n \geq k$, there is a vertex y in $V(G') \setminus D'$ that has a neighbor in the component of $G'[D']$ that contains x . The set $D = (D' \setminus \{x\}) \cup \{y\}$ is a k -component dominating set of G , which implies $\gamma_k(n) \leq \gamma_k(n + 1)$ as above.

(ii) This follows immediately from the trivial fact that every k -component dominating set is a k' -component dominating set. \square

Lemma 5 *If k is a positive integer, then $\gamma_k(2k+3) = k$.*

Proof: Since $\gamma_k(2k+3) \geq k$ follows immediately from the definition, it remains to show $\gamma_k(2k+3) \leq k$, which we prove by induction on k . Since every maximal outerplanar graph of order 5 has a universal vertex, we obtain $\gamma_1(5) \leq 1$. Now, let $k \geq 2$. Let G be a maximal outerplanar graph of order $2k+3$. Let x be a vertex of degree 2 in G . The neighbors of x in G , say u and v , are adjacent. Let G' arise from G by removing x and contracting the edge uv to a new vertex u^* . The order of G' is $2(k-1)+3$. By induction, G' has a $(k-1)$ -component dominating set D' of order $k-1$. If $u^* \in D'$, then let $D = (D' \setminus \{u^*\}) \cup \{u, v\}$. If $u^* \notin D'$, then let D arise from D' by adding one vertex from $\{u, v\}$ that has a neighbor in D' . Note that D is well defined because D' is a dominating set of G' . In both cases, D is a k -component dominating set of G of order k , which completes the proof. \square

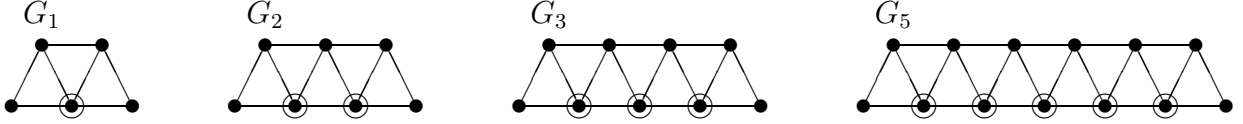


Figure 2: Some elements of a sequence $(G_k)_{k \in \mathbb{N}}$ of maximal outerplanar graphs of order $2k+3$. Considering the two vertices of degree 2 in the graph $G_{k+\ell}$ for positive integers k and ℓ with $\ell \leq k$, it follows easily that the encircled $k+\ell$ vertices form a minimum k -component dominating set of $G_{k+\ell}$.

Lemma 6 *If k is a positive integer and $n \in [2k+1, 4k+3]$, then $\gamma_k(n) = \lfloor \frac{kn}{2k+1} \rfloor$.*

Proof: By definition and Lemmas 4 and 5, we have

$$k = \left\lfloor \frac{k(2k+1)}{2k+1} \right\rfloor \leq \gamma_k(2k+1) \leq \gamma_k(2k+2) \leq \gamma_k(2k+3) = \left\lfloor \frac{k(2k+3)}{2k+1} \right\rfloor = k,$$

which implies the statement for $2k+1 \leq n \leq 2k+3$.

Now, let $n \in [2k+4, 4k+3]$. If n is odd, say $n = 2(k+\ell)+3$ for some positive integer ℓ with $\ell \leq k$, then the graph $G_{k+\ell}$ illustrated in Figure 2 easily implies $\gamma_k(n) \geq \gamma_k(G_{k+\ell}) = k+\ell$. If n is even, say $n = 2(k+\ell)+2$ for some positive integer ℓ with $\ell \leq k$, then the graph $G'_{k+\ell}$ that arises from $G_{k+\ell}$ by removing one vertex of degree 2 easily implies $\gamma_k(n) \geq \gamma_k(G'_{k+\ell}) = k+\ell$. Since

$$k+\ell \leq \frac{k(2(k+\ell)+2)}{2k+1} < \frac{k(2(k+\ell)+3)}{2k+1} < k+\ell+1$$

for $\ell \in [k]$, this implies $\gamma_k(n) \geq \lfloor \frac{kn}{2k+1} \rfloor$.

For $\ell \in [k]$, Lemmas 4 and 5 imply

$$\gamma_k(2(k+\ell)+2) \leq \gamma_k(2(k+\ell)+3) \leq \gamma_{k+\ell}(2(k+\ell)+3) = k+\ell,$$

which implies $\gamma_k(n) \leq \lfloor \frac{kn}{2k+1} \rfloor$, and completes the proof. \square

It is an immediate consequence of Lemmas 5 and 6 that for a positive integer k and a non-negative integer ℓ with $\ell \leq k$, we have

$$\gamma_k(2(k + \ell) + 2) = \gamma_k(2(k + \ell) + 3) = k + \ell. \quad (1)$$

Lemma 7 *Let k and n be positive integers, and let G be a maximal outerplanar graph of order n . Let u be a vertex of G , and let xy be a $C(G)$ -edge.*

- (i) *If $n = 2k + 1$, then G has a k -component dominating set D of order k that contains u .*
- (ii) *If $n = 2k + 2$, then G has a k -component dominating set D of order k that intersects xy .*
- (iii) *If $n = 2k + 1$, and x and y both have degree at least 3 in G , then G has a k -component dominating set D of order k that contains x and y .*
- (iv) *If $n = 2k + 2$, and x has degree at least 3 in G , then G has a k -component dominating set D of order k that contains x .*

Proof: Since the statements are trivial for $k = 1$, we consider $k \geq 2$.

(i) Since $2k + 1 = 2(k - 1) + 3$, Lemma 5 implies that G has $(k - 1)$ -component dominating set D' of order $k - 1$. If $u \notin D'$, then let $D = D' \cup \{u\}$. If $u \in D'$, then let $D = D' \cup \{v\}$ where $v \in V(G) \setminus D'$. In both cases, D has the desired properties.

(ii) Let G' arise from G by contracting the edge xy to a new vertex u^* . Since G' has order $2k + 1 = 2(k - 1) + 3$, Lemma 5 implies that G' has $(k - 1)$ -component dominating set D' of order $k - 1$. If $u^* \in D'$, then let $D = (D' \setminus \{u^*\}) \cup \{x, y\}$. If $u^* \notin D'$, then let D arise from D' by adding one vertex from $\{x, y\}$ that has a neighbor in D' . In both cases, D has the desired properties.

(iii) Let z be a vertex of G such that xyz is a triangle in G . Let G_x be the subgraph of G generated by the chord yz that does not contain x , and let G_y be the subgraph of G generated by the chord xz that does not contain y . Let G_x and G_y have orders $\ell_x + 1$ and $\ell_y + 1$, respectively. Note that $n = \ell_x + \ell_y + 1$, which implies that ℓ_x and ℓ_y have the same parity modulo 2.

If ℓ_x and ℓ_y are both even, then (i) implies that G_x has a $\ell_x/2$ -component dominating set D_x of order $\ell_x/2$ that contains y , and G_y has a $\ell_y/2$ -component dominating set D_y of order $\ell_y/2$ that contains x . Since $k = \frac{n-1}{2} = \ell_x/2 + \ell_y/2$, possibly adding one vertex to the set $D_x \cup D_y$ yields a set with the desired properties.

If ℓ_x and ℓ_y are both odd, then (ii) implies that G_x has a $(\ell_x - 1)/2$ -component dominating set D_x of order $(\ell_x - 1)/2$ that intersects yz , and G_y has a $(\ell_y - 1)/2$ -component dominating set D_y of order $(\ell_y - 1)/2$ that intersects xz . The set $D = D_x \cup D_y \cup \{x, y\}$ is a dominating set of G such that $G[D]$ is connected and $|D| \leq |D_x| + |D_y| + 1 = k$. Possibly adding further vertices to D yields a set with the desired properties.

(iv) If y has degree 2 in G , then $G' = G - y$ is a maximal outerplanar graph of order $2k + 1$. By (i), G' has a k -component dominating set D' of order k that contains x . Clearly, D' is also

a k -component dominating set of G . Hence, we may assume that y has degree at least 3 in G . Let z , G_x , G_y , ℓ_x , and ℓ_y be as in (iii). Since n is even, ℓ_x and ℓ_y have different parities modulo 2.

If ℓ_x is odd and ℓ_y is even, then (i) implies that G_y has a $\ell_y/2$ -component dominating set D_y of order $\ell_y/2$ that contains x , and (ii) implies that G_x has a $(\ell_x - 1)/2$ -component dominating set D_x of order $(\ell_x - 1)/2$ that intersects yz . Since $(\ell_x + \ell_y - 1)/2 = (n - 2)/2 = k$, possibly adding one further vertex to $D_x \cup D_y$ yields a set D with the desired properties.

If ℓ_x is even and ℓ_y is odd, then (ii) implies that G_y has a $(\ell_y - 1)/2$ -component dominating set D_y of order $(\ell_y - 1)/2$ that intersects xz , and (i) implies that G_x has a $\ell_x/2$ -component dominating set D_x of order $\ell_x/2$ that contains z . If $x \in D_y$, then possibly adding one vertex to $D_x \cup D_y$ yields a set D with the desired properties. If $x \notin D_y$, then $z \in D_x \cap D_y$, and $D = D_x \cup D_y \cup \{x\}$ has the desired properties. \square

For an even integer ℓ at least 4, let \mathcal{G}_ℓ be the set of all pairs (G, xy) , where

- G is a maximal outerplanar graph of order $\ell + 1$,
- xy is a $C(G)$ -edge such that $\{d_G(x), d_G(y)\} = \{2, 3\}$, and
- if $N_{C(G)}(x) = \{x', y\}$ and $N_{C(G)}(y) = \{y', x\}$, then the maximal outerplanar graph $G^- = G - \{x, y\}$ does not have a $(\ell/2 - 2)$ -component dominating set of order $\ell/2 - 2$ that intersects $x'y'$.

See Figure 3 for an illustration. In fact, generalizing the first two graphs in this figure in the obvious way implies that \mathcal{G}_ℓ is non-empty for every even ℓ at least 4.

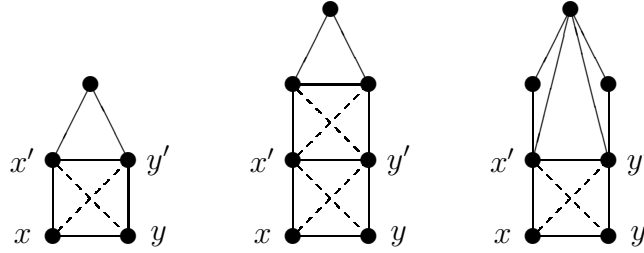


Figure 3: All graphs G for which (G, xy) belongs to \mathcal{G}_4 and \mathcal{G}_6 . From each pair of crossing dashed edges exactly one edge belongs to G .

For positive integers k and p with $p \leq k - 1$, let \mathcal{H}_k^p be the set of all graphs G such that there are

- $2p + 1$ even integers $\ell_1, \dots, \ell_{2p+1}$ with $4 \leq \ell_i \leq 2k$ for $i \in [2p + 1]$ and

$$\ell_1 + \dots + \ell_{2p+1} \geq 4kp + 2p + 2,$$

as well as

- $2p + 1$ pairs

$$(G_1, x_1y_1), \dots, (G_{2p+1}, x_{2p+1}y_{2p+1})$$

with $(G_i, x_iy_i) \in \mathcal{G}_{\ell_i}$ for $i \in [2p + 1]$ such that

G arises from the disjoint union of G_1, \dots, G_{2p+1} by

- identifying the two vertices y_i and x_{i+1} for every $i \in [2p + 1]$, where indices are identified modulo $2p + 1$, and
- triangulating the cycle $C_0(G) : x_1x_2 \dots x_{2p+1}x_1$.

The graphs in \mathcal{H}_k^p have a natural embedding illustrated in Figure 4. In what follows, we always assume the graphs in \mathcal{H}_k^p to be embedded in this way.

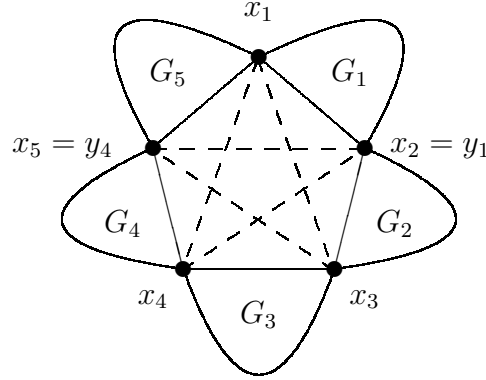


Figure 4: The embedding of a graph G in \mathcal{H}_k^2 . The dashed edges indicate a triangulation of the cycle $C_0(G) : x_1x_2x_3x_4x_5x_1$.

Let

$$\mathcal{H}_k = \bigcup_{p=1}^{k-1} \mathcal{H}_k^p.$$

Since for $k = 1$, there is no positive integer p with $p \leq k - 1$, the set \mathcal{H}_1 is empty. By definition, $\mathcal{H}_2 = \mathcal{H}_2^1$. In view of the unique element of \mathcal{G}_4 illustrated in Figure 3, the two graphs in Figure 1 form the only elements of \mathcal{H}_2 .

Lemma 8 *Let k and p be positive integers with $p \leq k - 1$. Let G be a graph in \mathcal{H}_k^p , and let ℓ_i and (G_i, x_iy_i) for $i \in [2p + 1]$ be as above.*

- (i) G has even order n at least $4kp + 2p + 2$ and at most $2k(2p + 1)$.
- (ii) $\ell_i + \ell_{i+1} \geq 2k + 2p + 2$ for every $i \in [2p + 1]$.
- (iii) $\gamma_k(G) = n/2 - p = \lceil \frac{kn}{2k+1} \rceil$.
- (iv) For every vertex u in $V(G) \setminus V(C_0(G))$, the graph G has a dominating set D of order at most $n/2 - (p + 1)$ such that

- D has a partition into two disjoint sets D_1 and D_2 ,
- $G[D_1]$ is a connected graph of order at least $\frac{\min\{\ell_i: i \in [2p+1]\}}{2} - 1$ that contains u ,
- every component of $G[D_2]$ has order at least k , and
- D contains no neighbor of u on $C_0(G)$.

Proof: (i) The lower bound on n and its parity modulo 2 are part of the definition of \mathcal{H}_k^p . Since each ℓ_i is at most $2k$, the upper bound follows immediately.

(ii) If there is some index $i \in [2p+1]$ with $\ell_i + \ell_{i+1} < 2k + 2k + 2$, then, since, each of the remaining $2p-1$ values ℓ_j is at most $2k$, we obtain $n < 2k + 2p + 2 + (2p-1)2k = 4kp + 2p + 2$, which contradicts (i).

(iii) Note that $n/2 - p = \lceil \frac{kn}{2k+1} \rceil$ is equivalent to $0 = \lceil \frac{1}{2k+1}((2k+1)p - n/2) \rceil$, which is equivalent to $-2k \leq (2k+1)p - n/2 \leq 0$. Therefore, the equality $n/2 - p = \lceil \frac{kn}{2k+1} \rceil$ follows easily from (i).

By Lemma 7(i), for every $i \in [2p+1]$, the graph G_i has a $\ell_i/2$ -component dominating set D_i^x of order $\ell_i/2$ that contains x_i as well as a $\ell_i/2$ -component dominating set D_i^y of order $\ell_i/2$ that contains $y_i = x_{i+1}$. By (ii), the set

$$D_1^y \cup D_2^x \cup D_3^y \cup D_4^x \cup \dots \cup D_{2p-1}^y \cup D_{2p}^x \cup D_{2p+1}^y$$

is a k -component dominating set of G of order at most $n/2 - p$, which implies $\gamma_k(G) \leq n/2 - p$.

It remains to show $\gamma_k(G) \geq n/2 - p$. Therefore, let D be a k -component dominating set of minimum order of G such that $|D \cap V(C_0(G))|$ is as large as possible. For $i \in [2p+1]$, let $D_i = D \cap (V(G_i) \setminus \{x_i, y_i\})$, $N_{C(G_i)}(x_i) = \{x'_i, y_i\}$, and $N_{C(G_i)}(y_i) = \{x_i, y'_i\}$.

If there is some $i \in [2p+1]$ such that $|D_i| \leq \ell_i/2 - 2 < k$, then, since D is a k -component dominating set of G , the set D_i intersects $x'_i y'_i$. This implies that D_i is a subset of some $(\ell_i/2 - 2)$ -component dominating set of the graph $G_i^- = G_i - \{x_i, y_i\}$ that is of order $\ell_i/2 - 2$ and intersects $x'_i y'_i$, which implies the contradiction $(G_i, x_i y_i) \notin \mathcal{G}_{\ell_i}$. Hence $|D_i| \geq \ell_i/2 - 1$ for every $i \in [2p+1]$.

If there is some $i \in [2p+1]$ such that $x_i, x_{i+1} \notin D$, then, since D is a k -component dominating set of G , the set D_i has at least $k \geq \ell_i/2$ elements. By Lemma 7 (i), the graph G_i has a k -component dominating set D'_i of order k that contains x_i . Now, $D' = (D \setminus D_i) \cup D'_i$ is a k -component dominating set of G such that $|D'| \leq |D|$ and $|D' \cap V(C_0(G))| > |D \cap V(C_0(G))|$, which contradicts the choice of D . Hence, for every $i \in [2p+1]$, we have $|D \cap \{x_i, x_{i+1}\}| \geq 1$, which implies $|D \cap V(C_0(G))| \geq p+1$.

Altogether, we obtain

$$|D| \geq \sum_{i=1}^{2p+1} (\ell_i/2 - 1) + (p+1) = n/2 - (2p+1) + (p+1) = n/2 - p,$$

which completes the proof of (iii).

(iv) By symmetry, we may assume that $u \in V(G_{2p+1})$. The graph $G_{2p+1}^- = G_{2p+1} - \{x_{2p+1}, y_{2p+1}\}$ has order $\ell_{2p+1} - 1$. By Lemma 7 (i), G_{2p+1}^- has a $(\ell_{2p+1}/2 - 1)$ -component dominating set D_{2p+1}^- of order $\ell_{2p+1}/2 - 1$ that contains u . Let D_i^x and D_i^y be as in (iii). Let $D = D_1 \cup D_2$, where $D_1 = D_{2p+1}^-$ and $D_2 = D_1^y \cup D_2^x \cup D_3^y \cup D_4^x \cup \dots \cup D_{2p-1}^y \cup D_{2p}^x$. Clearly, D is a dominating set of G of order at most $n/2 - (p+1)$, D_1 and D_2 are disjoint, $G[D_1]$ is a connected graph of order $\ell_{2p+1}/2 - 1$ that contains u , and every component of $G[D_2]$ has order at least k . If D contains a neighbor, say v , of u on $C_0(G)$, then $v \in D_2$, and it follows that the component of $G[D]$ that contains u has order at least k , which implies the contradiction that D is a k -component dominating set of order less than $n/2 - p$. Hence, D has the desired properties. \square

If G , u , and D are as in Lemma 8(iv), then D is a *semi- k -component dominating set of G with u in the small component*.

Lemma 9 (Shermer [8]) *Let s and n be positive integers with $s \geq 2$, and let G be a maximal outerplanar graph of order n . If $n \geq 2s$, then G has a chord xy such that one of the subgraphs of G generated by xy has m $C(G)$ -edges where $s \leq m \leq 2s - 2$.*

For a proof of Lemma 9, the reader may refer to [1, 8].

Lemma 10 *Let k and ℓ be positive integers with $2 \leq \ell \leq k$. If G is a maximal outerplanar graph of order $n = 4k + 2\ell$ that does not belong to \mathcal{H}_k , then $\gamma_k(G) \leq 2k + \ell - 2 = \lfloor \frac{kn}{2k+1} \rfloor$.*

Proof: Note that $2k + \ell - 2 = \lfloor \frac{kn}{2k+1} \rfloor$ is equivalent to $0 = \lfloor \frac{2k-\ell+2}{2k+1} \rfloor$, which is equivalent to $0 \leq 2k - \ell + 2 \leq 2k$. Therefore, the equality $2k + \ell - 2 = \lfloor \frac{kn}{2k+1} \rfloor$ follows easily from $n = 4k + 2\ell$ and $2 \leq \ell \leq k$.

It remains to show $\gamma_k(G) \leq 2k + \ell - 2$. For a contradiction, suppose that G is a graph of order $4k + 2\ell$ that does not belong to \mathcal{H}_k and satisfies $\gamma_k(G) > 2k + \ell - 2$. Since $n \geq 2(2k+2)$, Lemma 9 implies that G has a chord xy such that one of the subgraphs of G generated by xy , say G_{xy} , has m $C(G)$ -edges where $2k + 2 \leq m \leq 4k + 2$. We assume that xy is chosen such that m is smallest possible subject to these conditions. Let G_z be the subgraph of G generated by xy that is distinct from G_{xy} .

If $m = 2k + 2$, then G_{xy} has order $2k + 3$, and G_z has odd order $n - (2k + 1)$. By Lemma 5, Lemma 8(i), and the choice of G , the graph G_{xy} has a k -component dominating set of order k , and G_z has a k -component dominating set of order at most $\lfloor \frac{k(n-(2k+1))}{2k+1} \rfloor = k + \ell - 2$ whose union is a k -component dominating set of G of order at most $2k + \ell - 2$, which is a contradiction. Hence, $m > 2k + 2$.

Let z be the vertex of G_{xy} such that xyz is a triangle of G . Let G_x be the subgraph of G generated by yz that does not contain x , and let G_y be the subgraph of G generated by xz that does not contain y . Let G_x and G_y have orders $\ell_x + 1$ and $\ell_y + 1$, respectively. Let G_z have order $\ell_z + 1$. Note that $m = \ell_x + \ell_y$ and $n = \ell_x + \ell_y + \ell_z$. The choice of xy and $m > 2k + 2$ imply $\ell_x, \ell_y \geq 2$.

We consider different cases.

Case 1 ℓ_x and ℓ_y are both odd.

By Lemma 7(ii), G_x has a $(\ell_x - 1)/2$ -component dominating set D_x of order $(\ell_x - 1)/2$ that intersects yz , and G_y has a $(\ell_y - 1)/2$ -component dominating set D_y of order $(\ell_y - 1)/2$ that intersects xz . Note that $G[D_x \cup D_y]$ is a connected graph of order at least $(\ell_x - 1)/2 + (\ell_y - 1)/2 - 1 = m/2 - 2 \geq k$. Since m is even, the order of G_z is odd.

Suppose that D_x and D_y both contain z , which implies $|D_x \cup D_y| = m/2 - 2$. By Lemma 7(i), G_z has a $\ell_z/2$ -component dominating set D_z of order $\ell_z/2$ that contains x , and $D_x \cup D_y \cup D_z$ is a k -component dominating set of G of order at most $m/2 - 2 + \ell_z/2 = n/2 - 2 = 2k + \ell - 2$, which is a contradiction. Hence, D_x and D_y do not both contain z . By symmetry, we may assume that D_y contains x . Again, by Lemma 7(i), G_z has a $\ell_z/2$ -component dominating set D_z of order $\ell_z/2$ that contains x , and $D_x \cup D_y \cup D_z$ is a k -component dominating set of G of order at most $m/2 - 1 + \ell_z/2 - 1 = 2k + \ell - 2$, which is a contradiction.

Case 2 ℓ_x is odd and ℓ_y is even.

By Lemma 7(ii), G_x has a $(\ell_x - 1)/2$ -component dominating set D_x of order $(\ell_x - 1)/2$ that intersects yz . Suppose that D_x contains z . By Lemma 7(i), G_y has a $\ell_y/2$ -component dominating set D_y of order $\ell_y/2$ that contains z . Note that $G[D_x \cup D_y]$ is a connected graph of order $(\ell_x - 1)/2 + \ell_y/2 - 1 = (m - 3)/2 \geq k$. Since m is odd, the order of G_z is even. By Lemma 7(ii), G_z has a $(\ell_z - 1)/2$ -component dominating set D_z of order $(\ell_z - 1)/2$ that intersects xy , and $D_x \cup D_y \cup D_z$ is a k -component dominating set of G of order at most $(m - 3)/2 + (\ell_z - 1)/2 = n/2 - 2 = 2k + \ell - 2$, which is a contradiction. Hence, D_x contains y but not z . By Lemma 7(i), G_y has a $\ell_y/2$ -component dominating set D_y of order $\ell_y/2$ that contains x . Since D_z contains x or y , the set $D_x \cup D_y \cup D_z$ is a k -component dominating set of G of order at most $(m - 1)/2 + (\ell_z - 1)/2 - 1 = 2k + \ell - 2$, which is a contradiction.

Case 3 ℓ_x and ℓ_y are both even.

By Lemma 7(i), G_x has a $\ell_x/2$ -component dominating set D_x of order $\ell_x/2$ that contains z , and G_y has a $\ell_y/2$ -component dominating set D_y of order $\ell_y/2$ that contains z . Note that $G[D_x \cup D_y]$ is a connected graph of order $\ell_x/2 + \ell_y/2 - 1 = m/2 - 1 > k$. Since m is even, the order of G_z is odd. By the choice of xy , we have $\ell_x \leq 2k$ and $\ell_y \leq 2k$, which implies $\ell_y \geq 4$ and $\ell_x \geq 4$. If $m = 4k + 2$, then the choice of xy implies $\ell_x = \ell_y = 2k + 1$, which is a contradiction. Hence $m \leq 4k$, which implies that $\ell_z \geq 4$.

Let $N_{C_0(G_x)}(y) = \{y', z\}$ and $N_{C_0(G_x)}(z) = \{y, z'\}$.

Suppose that y and z both have degree at least 3 in G_x . By Lemma 7(iii), G_x has a $\ell_x/2$ -component dominating set D_x of order $\ell_x/2$ that contains y and z . By Lemma 7(i), G_y has a $\ell_y/2$ -component dominating set D_y of order $\ell_y/2$ that contains z , and G_z has a $\ell_z/2$ -component dominating set D_z of order $\ell_z/2$ that contains y . Since $\ell_x/2 + \ell_y/2 + \ell_z/2 - 2 = 2k + \ell - 2$, the set $D_x \cup D_y \cup D_z$ is a k -component dominating set of G of order at most $2k + \ell - 2$, which is a contradiction. Hence, by symmetry, we may assume that y has degree 2 in G_x , which implies that z is adjacent to y' .

Suppose that z has degree at least 4 in G_x . By Lemma 7(iv), $G_x - y$ has a $(\ell_x/2 - 1)$ -component dominating set D'_x of order $\ell_x/2 - 1$ that contains z . Choosing D_y and D_z as above, we obtain that $D_x \cup D_y \cup D_z$ is a k -component dominating set of G of order at most $2k + \ell - 2$, which is a contradiction. Hence, we may assume that z has degree 3 in G_x .

By symmetry, we obtain that $\{d_{G_x}(y), d_{G_x}(z)\} = \{d_{G_y}(x), d_{G_y}(z)\} = \{2, 3\}$. Furthermore, since our argument did not use the fact that $\ell_x \leq 2k$, we also obtain, by symmetry, that $\{d_{G_z}(x), d_{G_z}(y)\} = \{2, 3\}$.

Suppose that $\ell_z \geq 2k + 2$. By Lemma 7(i), G_x has a $\ell_x/2$ -component dominating set D_x of order $\ell_x/2$ that contains z , G_y has a $\ell_y/2$ -component dominating set D_y of order $\ell_y/2$ that contains z , and $G_z^- = G_z - \{x, y\}$ has a $(\ell_z/2 - 1)$ -component dominating set D_z^- of order $\ell_z/2 - 1 \geq k$. Now, $D_x \cup D_y \cup D_z^-$ is a k -component dominating set of G of order $2k + \ell - 2$, which is a contradiction. Hence, $\ell_z \leq 2k$.

Suppose that (G_x, yz) does not lie in \mathcal{G}_{ℓ_x} . Since the order of G_x is $\ell_x + 1$, the definition of \mathcal{G}_{ℓ_x} implies the existence of a $(\ell_x/2 - 2)$ -component dominating set D_x^- of $G_x^- = G_x - \{y, z\}$ that intersects $y'z'$. By Lemma 7(i), G_y has a $\ell_y/2$ -component dominating set D_y of order $\ell_y/2$ that contains z , and G_z has a $\ell_z/2$ -component dominating set D_z of order $\ell_z/2$ that contains y . Now, $D_x^- \cup D_y \cup D_z$ is a k -component dominating set of G of order at most $2k + \ell - 2$, which is a contradiction.

By symmetry, we obtain that $(G_x, yz) \in \mathcal{G}_{\ell_x}$, $(G_y, xz) \in \mathcal{G}_{\ell_y}$, and $(G_z, xy) \in \mathcal{G}_{\ell_z}$, which implies the contradiction $G \in \mathcal{H}_k^1 \subseteq \mathcal{H}_k$. \square

We proceed to the proof of our main result.

Proof of Theorem 3: Suppose for a contradiction, that G is a counterexample of minimum order n . Lemma 6 implies $n \geq 4k + 4$. Lemma 8 implies $G \notin \mathcal{H}_k$ and $\gamma_k(G) > \lfloor \frac{kn}{2k+1} \rfloor$.

Claim 1 $n \bmod (2k + 1) = 2\ell$ for some $\ell \in [k - 1]$.

Proof of Claim 1: Suppose for a contradiction that $n \bmod (2k + 1) \notin \{2\ell : \ell \in [k - 1]\}$.

Clearly, G contains no two adjacent vertices of degree 2. If G does not contain either two vertices u and v of degree 2 at distance 2 or two adjacent vertices u and v such that u has degree 2 and v has degree 3, then removing from G all vertices of degree 2 results in a maximal outerplanar graph of minimum degree at least 3, which is a contradiction. Hence, let u and v have the stated properties.

The graph $G' = G - \{u, v\}$ is a maximal outerplanar graph of order $n - 2$. In the first case, let $N_G(u) = \{x, y\}$ and $N_G(v) = \{x, z\}$. Note that xy and yz are edges of G' that belong to $C(G')$. In the second case, let $N_G(u) = \{v, x\}$ and $N_G(v) = \{u, x, y\}$. Note that xy is an edge of G' that belongs to $C(G')$. See Figure 5 for an illustration.

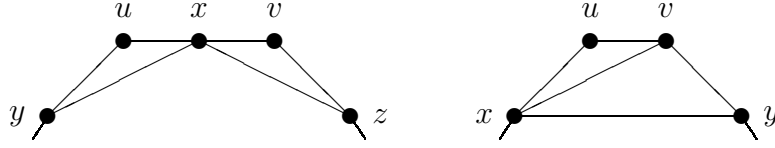


Figure 5: The two possibilities for u and v in the proof of the claim.

Suppose that $G' \notin \mathcal{H}_k$. By the choice of G , the graph G' has a k -component dominating set D' of order at most $\left\lfloor \frac{k(n-2)}{2k+1} \right\rfloor$, and $D = D' \cup \{x\}$ is a k -component dominating set of G of order at most $\left\lfloor \frac{k(n-2)}{2k+1} \right\rfloor + 1 = \left\lfloor \frac{kn+1}{2k+1} \right\rfloor$. Since $n \not\equiv 2 \pmod{2k+1}$, and k and $2k+1$ are coprime, we have $kn \not\equiv 2k \pmod{2k+1}$, which implies the contradiction $|D| \leq \left\lfloor \frac{kn+1}{2k+1} \right\rfloor = \left\lfloor \frac{kn}{2k+1} \right\rfloor$. Hence, $G' \in \mathcal{H}_k^p$ for some $p \in [k-1]$. By Lemma 8(i), $n = n' + 2$ is an even integer at least $4kp + 2p + 4 = 2p(2k+1) + 4$ and at most $2k(2p+1) + 2 = 2p(2k+1) + 2(k-p+1)$, which implies $n \pmod{2k+1} \in \{2\ell : \ell \in [2, k-p+1]\}$. Our assumption implies $n \pmod{2k+1} = 2k$, which implies $p = 1$ and $n = 2k(2p+1) + 2 = 6k + 2$. If G' arises as in the definition of \mathcal{H}_k^1 by suitably identifying vertices in three graphs from \mathcal{G}_{ℓ_1} , \mathcal{G}_{ℓ_2} , and \mathcal{G}_{ℓ_3} , respectively, then $\ell_i \leq 2k$ and $n' = 6k$ imply $\ell_1 = \ell_2 = \ell_3 = 2k$.

If $x \notin V(C_0(G'))$, then Lemma 8(iv) implies that G' has a semi- k -component dominating set D' of order $\left\lfloor \frac{k(n-2)}{2k+1} \right\rfloor$ with x in the small component, which is of order $k-1$. In this case, let $D = D' \cup \{u\}$. If $x \in V(C_0(G'))$, then $y \notin V(C_0(G'))$, and Lemma 8(iv) implies that G' has a semi- k -component dominating set D' of order $\left\lfloor \frac{k(n-2)}{2k+1} \right\rfloor$ with $x \notin D'$ and y in the small component, which is of order $k-1$. In this case, let $D = D' \cup \{x\}$. In both cases, D is a k -component dominating set of G of order $\left\lfloor \frac{kn+1}{2k+1} \right\rfloor = \left\lfloor \frac{kn}{2k+1} \right\rfloor$, which is a contradiction. \square

Suppose $n \leq 6k + 4$. Since $n \geq 4k + 4$, the claim implies $n = 2(2k+1) + 2\ell = 4k + 2(\ell+1)$ for some $\ell \in [k-1]$. By Lemma 10, $\gamma_k(G) \leq \left\lfloor \frac{kn}{2k+1} \right\rfloor$, which is a contradiction. Hence, $n \geq 6k + 5$.

Since $6k + 5 \geq 2(2k+2)$, Lemma 7 implies the existence of a chord xy such that one of the subgraphs of G generated by xy , say G_{xy} , has m $C(G)$ -edges where $2k+2 \leq m \leq 4k+2$. We assume that xy is chosen such that m is smallest possible subject to these conditions. Let G_z denote the subgraph of G generated by xy distinct from G_{xy} . Note that G_z has order $n - m + 1 \geq (6k+5) - (4k+2) + 1 = 2k+4$, that is, contracting one or two edges of G_z yields a graph of order at least $2k+2$.

Suppose $m = 2k+2$. If $G_z \notin \mathcal{H}_k$, then Lemma 5 and the choice of G imply

$$\gamma_k(G) \leq \gamma_k(G_{xy}) + \gamma_k(G_z) \leq k + \left\lfloor \frac{k(n - (2k+1))}{2k+1} \right\rfloor = \left\lfloor \frac{kn}{2k+1} \right\rfloor,$$

which is a contradiction. Hence, $G_z \in \mathcal{H}_k$. Since xy is an edge of $C_0(G_z)$, we obtain that either x or y does not belong to $C_0(G_z)$. By symmetry, we may assume that x does not belong to $C_0(G_z)$. By Lemma 8(iv), G_z has a semi- k -component dominating set D_z of order $\left\lfloor \frac{k(n-(2k+1))}{2k+1} \right\rfloor$ with x in the small component. By Lemma 7(i), G_{xy} has a $(k+1)$ -component dominating set D_{xy} of order $k+1$ that contains x . Now, $D_{xy} \cup D_z$ is a k -component dominating set of G of order at most $(k+1) + \left\lfloor \frac{k(n-(2k+1))}{2k+1} \right\rfloor - 1 = \left\lfloor \frac{kn}{2k+1} \right\rfloor$, which is a contradiction. Hence, $m > 2k+2$.

Let z be the vertex of G_{xy} such that xyz is a triangle of G . Let G_x be the subgraph of G generated by yz that does not contain x , and let G_y be the subgraph of G generated by xz that does not contain y . Let G_x and G_y have orders $\ell_x + 1$ and $\ell_y + 1$, respectively. The choice of xy and $m > 2k + 2$ imply $\ell_x, \ell_y \geq 2$. Let G_z have order $\ell_z + 1$. Note that $m = \ell_x + \ell_y$ and $n = \ell_x + \ell_y + \ell_z$.

We consider different cases.

Case 1 ℓ_x and ℓ_y are both odd.

By Lemma 7(i), G_x has a $(\ell_x - 1)/2$ -component dominating set D_x of order $(\ell_x - 1)/2$ that intersects yz , and G_y has a $(\ell_y - 1)/2$ -component dominating set D_y of order $(\ell_y - 1)/2$ that intersects xz . Since $m \geq 2k + 4$, $G[D_x \cup D_y]$ is a connected graph of order at least $m/2 - 2 \geq k$.

Case 1.1 $z \in D_x \cap D_y$.

Suppose $G_z \notin \mathcal{H}_k$. We obtain that $\gamma_k(G)$ is at most

$$|D_x \cup D_y| + \gamma_k(G_z) \leq m/2 - 2 + \left\lfloor \frac{k(n - m + 1)}{2k + 1} \right\rfloor = \left\lfloor \frac{kn}{2k + 1} + \frac{m/2 - 3k - 2}{2k + 1} \right\rfloor \leq \left\lfloor \frac{kn}{2k + 1} \right\rfloor,$$

where the last inequality is implied by $m < 6k + 4$, which is a contradiction. Hence, $G_z \in \mathcal{H}_k$.

By symmetry, we may assume that x does not belong to $C_0(G_z)$. By Lemma 8(iv), G_z has a semi- k -component dominating set D_z of order $\left\lfloor \frac{k(n - m + 1)}{2k + 1} \right\rfloor$ with x in the small component.

Again, $\gamma_k(G) \leq m/2 - 2 + \left\lfloor \frac{k(n - m + 1)}{2k + 1} \right\rfloor \leq \left\lfloor \frac{kn}{2k + 1} \right\rfloor$, which is a contradiction.

Case 1.2 $z \notin D_x \cap D_y$.

By symmetry, we may assume that $y \in D_x$. Let G'_z arise from G_z by contracting the edge xy to a new vertex u^* . Suppose $G'_z \notin \mathcal{H}_k$. By the choice of G , the graph G'_z has a k -component dominating set D'_z of order at most $\left\lfloor \frac{k(n - m)}{2k + 1} \right\rfloor$. If $u^* \notin D'_z$, then let $D = D_x \cup D_y \cup D'_z$. If $u^* \in D'_z$, then let $D = D_x \cup D_y \cup (D'_z \setminus \{u^*\}) \cup \{x\}$. In both cases, D is a k -component dominating set of G of order at most

$$m/2 - 1 + \left\lfloor \frac{k(n - m)}{2k + 1} \right\rfloor = \left\lfloor \frac{kn}{2k + 1} + \frac{m/2 - 2k - 1}{2k + 1} \right\rfloor \leq \left\lfloor \frac{kn}{2k + 1} \right\rfloor,$$

where the last inequality is implied by $m \leq 4k + 2$, which is a contradiction. Hence, $G'_z \in \mathcal{H}_k$.

Suppose $u^* \notin C_0(G'_z)$. By Lemma 8(iv), G'_z has a semi- k -component dominating set D'_z of order $\left\lfloor \frac{k(n - m)}{2k + 1} \right\rfloor$ with u^* in the small component. The set $D_x \cup D_y \cup (D'_z \setminus \{u^*\}) \cup \{x\}$ is a k -component dominating set of G of order at most $m/2 - 1 + \left\lfloor \frac{k(n - m)}{2k + 1} \right\rfloor \leq \left\lfloor \frac{kn}{2k + 1} \right\rfloor$, which is a contradiction.

Hence, $u^* \in C_0(G'_z)$. If $N_{C(G_z)}(y) = \{x, y'\}$, then $y' \notin C_0(G'_z)$. By Lemma 8(iv), G'_z has a semi- k -component dominating set D'_z of order $\left\lfloor \frac{k(n - m)}{2k + 1} \right\rfloor$ with y' in the small component such that $u^* \notin D'_z$. The set $D_x \cup D_y \cup D'_z$ is a k -component dominating set of G of order at most $m/2 - 1 + \left\lfloor \frac{k(n - m)}{2k + 1} \right\rfloor \leq \left\lfloor \frac{kn}{2k + 1} \right\rfloor$, which is a contradiction.

Case 2 ℓ_x is odd and ℓ_y is even.

Note that m is odd, which implies $m \leq 4k + 1$. By Lemma 7(ii), G_x has a $(\ell_x - 1)/2$ -component dominating set D_x of order $(\ell_x - 1)/2$ that intersects yz .

Case 2.1 $z \in D_x$.

By Lemma 7(i), G_y has a $\ell_y/2$ -component dominating set D_y of order $\ell_y/2$ that contains z . Since $m \geq 2k + 3$, we have $|D_x \cup D_y| = m/2 - 3/2 \geq k$. Suppose $G_z \notin \mathcal{H}_k$. By the choice of G , the graph G_z has a k -component dominating set D_z of order at most $\left\lfloor \frac{k(n-m+1)}{2k+1} \right\rfloor$. Now, the set $D_x \cup D_y \cup D_z$ is a k -component dominating set of G of order at most

$$m/2 - 3/2 + \left\lfloor \frac{k(n-m+1)}{2k+1} \right\rfloor = \left\lfloor \frac{kn}{2k+1} + \frac{m/2 - 2k - 3/2}{2k+1} \right\rfloor \leq \left\lfloor \frac{kn}{2k+1} \right\rfloor,$$

where the last inequality is implied by $m < 4k + 3$, which is a contradiction. Hence, $G_z \in \mathcal{H}_k$. By Lemma 8(iv), G_z has a semi- k -component dominating set D_z of order $\left\lfloor \frac{k(n-m+1)}{2k+1} \right\rfloor$ with x or y in the small component. Again, the set $D_x \cup D_y \cup D_z$ is a k -component dominating set of G of order at most $m/2 - 3/2 + \left\lfloor \frac{k(n-m+1)}{2k+1} \right\rfloor \leq \left\lfloor \frac{kn}{2k+1} \right\rfloor$, which is a contradiction.

Case 2.2 $z \notin D_x$.

We have $y \in D_x$. By Lemma 7(i), G_y has a $\ell_y/2$ -component dominating set D_y of order $\ell_y/2$ that contains x . Note that $|D_x \cup D_y| = m/2 - 1/2 > k$. Let $N_{C(G_z)}(y) = \{x, y'\}$ and $N_{C(G_z)}(x) = \{x', y\}$. Let G_z'' arise from G_z by contracting the two edges xy and yy' to a new vertex u^* . Suppose $G_z'' \notin \mathcal{H}_k$. By the choice of G , the graph G_z'' has a k -component dominating set D_z'' of order at most $\left\lfloor \frac{k(n-m-1)}{2k+1} \right\rfloor$. If $u^* \in D_z''$, then let $D = D_x \cup D_y \cup (D_z'' \setminus \{u^*\}) \cup \{y'\}$. If $u^* \notin D_z''$, then let $D = D_x \cup D_y \cup D_z''$. The set D is a k -component dominating set of G of order at most

$$m/2 - 1/2 + \left\lfloor \frac{k(n-m-1)}{2k+1} \right\rfloor = \left\lfloor \frac{kn}{2k+1} + \frac{m/2 - 2k - 1/2}{2k+1} \right\rfloor \leq \left\lfloor \frac{kn}{2k+1} \right\rfloor$$

where the last inequality is implied by $m \leq 4k + 1$, which is a contradiction. Hence, $G_z'' \in \mathcal{H}_k$. Suppose $u^* \notin C_0(G_z'')$. By Lemma 8(iv), G_z'' has a semi- k -component dominating set D_z'' of order $\left\lfloor \frac{k(n-m-1)}{2k+1} \right\rfloor$ with u^* in the small component. The set $D_x \cup D_y \cup (D_z'' \setminus \{u^*\}) \cup \{y'\}$ is a k -component dominating set of G of order at most $m/2 - 1/2 + \left\lfloor \frac{k(n-m-1)}{2k+1} \right\rfloor \leq \left\lfloor \frac{kn}{2k+1} \right\rfloor$, which is a contradiction. Hence, $u^* \in C_0(G_z'')$. By Lemma 8(iv), G_z'' has a semi- k -component dominating set D_z'' of order $\left\lfloor \frac{k(n-m-1)}{2k+1} \right\rfloor$ with x' in the small component such that $u^* \notin D_z''$. The set $D_x \cup D_y \cup D_z''$ is a k -component dominating set of G of order at most $m/2 - 1/2 + \left\lfloor \frac{k(n-m-1)}{2k+1} \right\rfloor \leq \left\lfloor \frac{kn}{2k+1} \right\rfloor$, which is a contradiction.

Case 3 ℓ_x and ℓ_y are both even.

We have $m \geq 2k + 4$. By the choice of xy , this implies $4 \leq \ell_x, \ell_y \leq 2k$, and, hence, $m \leq 4k$. Let G_z' arise from G_z by contracting the edge xy to a new vertex u^* .

Suppose $G'_z \notin \mathcal{H}_k$. By the choice of G , the graph G'_z has a k -component dominating set D'_z of order at most $\left\lfloor \frac{k(n-m)}{2k+1} \right\rfloor$. If $u^* \in D'_z$, then Lemma 7(i) implies that G_x has a $\ell_x/2$ -component dominating set D_x of order $\ell_x/2$ that contains y , and G_y has a $\ell_y/2$ -component dominating set D_y of order $\ell_y/2$ that contains x . In this case, let $D = D_x \cup D_y \cup (D'_z \setminus \{u^*\})$. If $u^* \notin D'_z$, then Lemma 7(i) implies that G_x has a $\ell_x/2$ -component dominating set D_x of order $\ell_x/2$ that contains z , and G_y has a $\ell_y/2$ -component dominating set D_y of order $\ell_y/2$ that contains z . In this case, let $D = D_x \cup D_y \cup D'_z$. In both cases D is a k -component dominating set of G of order at most $m/2 - 1 + \left\lfloor \frac{k(n-m)}{2k+1} \right\rfloor \leq \left\lfloor \frac{kn}{2k+1} \right\rfloor$ (cf. the calculation in Case 1.2), which is a contradiction. Hence, $G'_z \in \mathcal{H}_k$.

Suppose $u^* \notin C_0(G'_z)$. By Lemma 7(i), G_x has a $\ell_x/2$ -component dominating set D_x of order $\ell_x/2$ that contains y , and G_y has a $\ell_y/2$ -component dominating set D_y of order $\ell_y/2$ that contains x . Note that $k < m/2 - 1 \leq |D_x \cup D_y| \leq m/2$. By Lemma 8(iv), G'_z has a semi- k -component dominating set D'_z of order $\left\lfloor \frac{k(n-m)}{2k+1} \right\rfloor$ with u^* in the small component. The set $D_x \cup D_y \cup (D'_z \setminus \{u^*\})$ is a k -component dominating set of G of order at most $m/2 - 1 + \left\lfloor \frac{k(n-m)}{2k+1} \right\rfloor \leq \left\lfloor \frac{kn}{2k+1} \right\rfloor$, which is a contradiction. Hence, $u^* \in C_0(G'_z)$.

Let $C_0(G'_z)$ have order $2p + 1$ for some $p \in [k - 1]$. For $i \in [2p + 1]$, let ℓ_i with $4 \leq \ell_i \leq 2k$ and $(G_i, x_i y_i) \in \mathcal{G}_{\ell_i}$ be as in the definition of \mathcal{H}_k^p such that G'_z arises by suitably identifying vertices in the graphs G_1, \dots, G_{2p+1} , that is, $C_0(G'_z)$ is the cycle $x_1 x_2 \dots x_{2p+1} x_1$. Let $u^* = x_1$, that is, $xzyx_2 \dots x_{2p+1} x$ is a cycle in G . For $i \in [2p + 1]$, let D_i^x and D_i^y be as in the proof of Lemma 8(iii).

By Lemma 7(i), G_x has a $\ell_x/2$ -component dominating set D_x of order $\ell_x/2$ that contains z , and G_y has a $\ell_y/2$ -component dominating set D_y of order $\ell_y/2$ that contains z . Now, the set

$$D_x \cup D_y \cup D_1^y \cup D_2^x \cup D_3^y \cup D_4^x \cup \dots \cup D_{2p-1}^y \cup D_{2p}^x \cup D_{2p+1}^y$$

is a k -component dominating set of G of order at most $n/2 - (p + 1)$. By the choice of G , we obtain $\left\lfloor \frac{kn}{2k+1} \right\rfloor < \gamma_k(G) \leq n/2 - (p + 1)$, which implies $n \geq 4k(p + 1) + 2(p + 1) + 2$.

Suppose that $p = k - 1$. We obtain $n \geq 4k(p + 1) + 2(p + 1) + 2 = 4k^2 + 2k + 2$ as well as $n \leq n(G'_z) + m \leq (2k - 1)2k + m \leq 4k^2 + 2k$, which is a contradiction. Hence, $p \leq k - 2$, which implies $k \geq 3$.

Suppose that $d_{G_y}(x), d_{G_y}(z) \geq 3$. By Lemma 7(iii), G_y has a $\ell_y/2$ -component dominating set D_y of order $\ell_y/2$ that contains x and z . Choosing D_x as above, the set

$$D_x \cup D_y \cup D_1^y \cup D_2^x \cup D_3^y \cup D_4^x \cup \dots \cup D_{2p-1}^y \cup D_{2p}^x \cup D_{2p+1}^y$$

is a k -component dominating set of G of order at most $n/2 - (p + 2)$. Since $n \leq (2p + 1)2k + 2k$, we obtain $\gamma_k(G) \leq n/2 - (p + 2) \leq \left\lfloor \frac{kn}{2k+1} \right\rfloor$, which is a contradiction. Hence, one of the two degrees $d_{G_y}(x)$ and $d_{G_y}(z)$ is 2.

Suppose that $d_{G_y}(x) = 2$ and $d_{G_y}(z) \geq 4$. By Lemma 7(iv), $G_y - x$ has a $(\ell_y/2 - 1)$ -component dominating set D_y of order $\ell_y/2 - 1$ that contains z . Choosing D_x as above, the

set

$$D_x \cup D_y \cup D_1^y \cup D_2^x \cup D_3^y \cup D_4^x \cup \dots \cup D_{2p-1}^y \cup D_{2p}^x \cup D_{2p+1}^y$$

is a k -component dominating set of G of order at most $n/2 - (p+2) \leq \lfloor \frac{kn}{2k+1} \rfloor$, which is a contradiction. Hence, if $d_{G_y}(x) = 2$, then $d_{G_y}(z) = 3$. By a symmetric argument, we obtain that if $d_{G_y}(z) = 2$, then $d_{G_y}(x) = 3$, that is, $\{d_{G_y}(x), d_{G_y}(z)\} = \{2, 3\}$. By symmetry, $\{d_{G_x}(y), d_{G_x}(z)\} = \{2, 3\}$.

Let $N_{C(G_y)}(x) = \{x', z\}$ and $N_{C(G_y)}(z) = \{x, z'\}$.

Suppose that (G_y, xz) does not belong to \mathcal{G}_{ℓ_y} . By the definition of \mathcal{G}_{ℓ_y} , this implies that the graph $G_y^- = G_y - \{x, z\}$ has a $(\ell_y/2 - 2)$ -component dominating set D_y^- of order $\ell_y/2 - 2$ that intersects $x'z'$. Let D_x be as above. Now, the set

$$D_x \cup D_y^- \cup D_1^y \cup D_2^x \cup D_3^y \cup D_4^x \cup \dots \cup D_{2p-1}^y \cup D_{2p}^x \cup D_{2p+1}^y$$

contains $x \in D_{2p+1}^y$ and $z \in D_x$, which implies that it is a k -component dominating set of G of order at most $n/2 - (p+2) \leq \lfloor \frac{kn}{2k+1} \rfloor$, which is a contradiction. Hence, $(G_y, xz) \in \mathcal{G}_{\ell_y}$, and, by symmetry, $(G_x, yz) \in \mathcal{G}_{\ell_x}$. Altogether, this implies that $G \in \mathcal{H}_k^{p+1}$, which is the final contradiction and completes the proof. \square

Let k and n be positive integers with $n \geq 2k+1$. Lemma 6 and Lemma 8(iii) imply that

$$\gamma_k(n) = \begin{cases} \lfloor \frac{kn}{2k+1} \rfloor, & \text{if } n \in [2k+1, 4k+3], \text{ and} \\ \lceil \frac{kn}{2k+1} \rceil, & \text{if } n \text{ is an even number in } [4k+4, 4k^2-2k]. \end{cases}$$

Figure 6 illustrates how to construct maximal outerplanar graphs G of arbitrary order n with $n \bmod (2k+1) = 0$ that satisfy $\gamma_k(G) = \lfloor \frac{kn}{2k+1} \rfloor$.

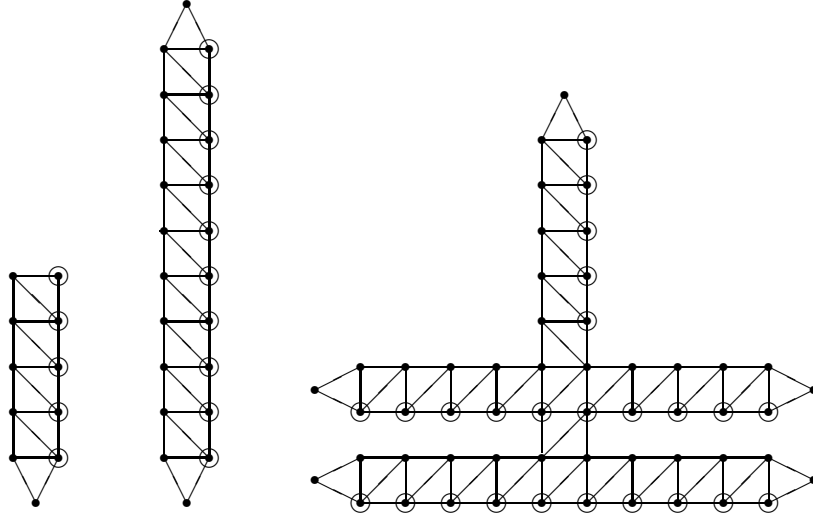


Figure 6: Graphs of order $s \cdot (2k+1)$ for $k=5$ and $s \in \{1, 2, 5\}$. Considering the vertices of degree 2, it follows easily that the encircled vertices form minimum k -component dominating sets.

If $n \in \{s \cdot (2k+1) + (2t-1), s \cdot (2k+1) + 2t\}$ for positive integers s and t with $t \in [k]$, then

$\lfloor \frac{kn}{2k+1} \rfloor = sk + t - 1$. The graphs in Figure 7 illustrate how to construct extremal maximal outerplanar graphs of these orders.

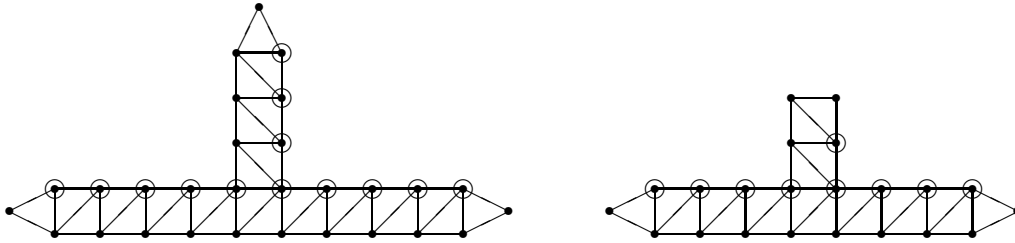


Figure 7: On the left, a graph G of order $n = s \cdot (2k + 1) + (2t - 1)$ for $k = 5$, $s = 2$, and $t = 4$. On the right, a graph G of order $n = s' \cdot (2k' + 1) + 2t'$ for $k' = 4$, $s' = 2$, and $t' = 2$. Again, the encircled vertices form minimum k -component dominating sets.

Altogether, it follows that

$$\gamma_k(n) = \begin{cases} \lceil \frac{kn}{2k+1} \rceil, & \text{if } n \text{ is an even number in } [4k + 4, 4k^2 - 2k] \text{ and} \\ \lfloor \frac{kn}{2k+1} \rfloor, & \text{otherwise.} \end{cases}$$

We introduced the parameter γ_k with the intention to obtain a common generalization of two separate results; one concerning the domination number and one concerning the total domination number. It seems interesting to unify/generalize further pairs of results about these parameters in this way.

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